we obtain an injective immersion $f: X \longrightarrow \mathbf{R}^{2k+1}$ such that $|f(x)| < 1$ for all $x \in X$. Let $\rho: X \longrightarrow \mathbf{R}$ be a proper function, and define a new injective immersion $F: X \longrightarrow \mathbf{R}^{2k+2}$ by $F(x) = (f(x), \rho(x))$. Now drop back down to $\mathbf{R}^{2k+1}$ as in the earlier theorem by composing F with an orthogonal projection $\pi: \mathbf{R}^{2k+2} \longrightarrow H$, where H is the linear space perpendicular to a suitable unit vector, a in $\mathbf{R}^{2k+2}$.

Recall that the map $\pi \circ F: X \longrightarrow H$ is still an injective immersion for almost every $a \in S^{2k+1}$, so we may pick an a that happens to be neither of the sphere's two poles. But now $\pi \circ F$ is easily seen to be proper. In fact, given any bound c, we claim that there exists another number d such that the set of points $x \in X$ where $|\pi \circ F(x)| \leq c$ is contained in the set where $|\rho(x)| \leq d$. As ρ is proper, the latter is a compact subset of X. Thus the claim implies that the preimage under $\pi \circ F$ of every closed ball in H is a compact subset of X, showing that $\pi \circ F$ is proper. If the claim is false, then there exists a sequence of points $\{x_i\}$ in X for which $|\pi \circ F(x_i)| < c$ but $\rho(x_i) \longrightarrow \infty$. Remember that, by definition, for every $z \in \mathbf{R}^{2k+2}$ the vector $\pi(z)$ is the one point in H for which $z - \pi(z)$ is a multiple of a. Thus $F(x_i) - \pi \circ F(x_i)$ is a multiple of a for each i, and hence so is the vector

$$w_i = \frac{1}{\rho(x_i)}[F(x_i) - \pi \circ F(x_i)].$$

Consider what happens as $i \longrightarrow \infty$.

$$\frac{F(x_i)}{\rho(x_i)} = \left(\frac{f(x_i)}{\rho(x_i)}, 1\right) \longrightarrow (0, \ldots, 0, 1),$$

because $|f(x_i)| < 1$ for all i. The quotient

$$\frac{\pi \circ F(x_i)}{\rho(x_i)}$$

has norm $\leq c/\rho(x_i)$, so it converges to zero. Thus $w_i \longrightarrow (0, \ldots, 0, 1)$. But each w_i is a multiple of a; therefore so is the limit. We conclude that a must be either the north or south pole of S^{2k+1}, a contradiction. This proves the claim and the theorem. Q.E.D.

EXERCISES

1. Show that $T(\mathbf{R}^k) = \mathbf{R}^k \times \mathbf{R}^k$.

2. Let g be a smooth, everywhere-positive function on X. Check that the multiplication map $T(X) \longrightarrow T(X)$, $(x, v) \longrightarrow (x, g(x)v)$, is smooth.

3. Show that $T(X \times Y)$ is diffeomorphic to $T(X) \times T(Y)$.

4. Show that the tangent bundle to S^1 is diffeomorphic to the cylinder $S^1 \times \mathbf{R}^1$.

5. Prove that the *projection* map $p: T(X) \longrightarrow X$, $p(x, v) = x$, is a submersion.

*6. A *vector field* $\vec{v}$ on a manifold X in $\mathbf{R}^N$ is a smooth map $\vec{v}: X \longrightarrow \mathbf{R}^N$ such that $\vec{v}(x)$ is always tangent to X at x. Verify that the following definition (which does not explicitly mention the ambient $\mathbf{R}^N$) is equivalent: a vector field $\vec{v}$ on X is a *cross section* of $T(X)$—that is, a smooth map $\vec{v}: X \longrightarrow T(X)$ such that $p \circ \vec{v}$ equals the identity map of X. (p as in Exercise 5.)

*7. A point $x \in X$ is a *zero* of the vector field $\vec{v}$ if $\vec{v}(x) = 0$. Show that if k is odd, there exists a vector field $\vec{v}$ on S^k having no zeros. [HINT: For $k = 1$, use $(x_1, x_2) \longrightarrow (-x_2, x_1)$.] It is a rather deep topological fact that nonvanishing vector fields do not exist on the even spheres. We will see why in Chapter 3.

*8. Prove that if S^k has a nonvanishing vector field, then its antipodal map is homotopic to the identity (Compare Section 6, Exercise 7.) [HINT: Show that you may take $|\vec{v}(x)| = 1$ everywhere. Now rotate x to $-x$ in the direction indicated by $\vec{v}(x)$.]

9. Let $S(X)$ be the set of points $(x, v) \in T(X)$ with $|v| = 1$. Prove that $S(X)$ is a $2k - 1$ dimensional submanifold of $T(X)$; it is called the *sphere bundle* of X. [HINT: Consider the map $(x, v) \longrightarrow |v|^2$.]

10. *The Whitney Immersion Theorem.* Prove that every k-dimensional manifold X may be immersed in $\mathbf{R}^{2k}$.

11. Show that if X is a compact k-dimensional manifold, then there exists a map $X \longrightarrow \mathbf{R}^{2k-1}$ that is an immersion except at finitely many points of X. Do so by showing that if $f: X \longrightarrow \mathbf{R}^{2k}$ is an immersion and a is a regular value for the map $F: T(X) \longrightarrow \mathbf{R}^{2k}$, $F(x, v) = df_x(v)$, then $F^{-1}(a)$ is a finite set. Show that $\pi \circ f$ is an immersion except on $f^{-1}(a)$, where π is an orthogonal projection perpendicular to a. The exceptional points, in $f^{-1}(a)$, are called *cross caps.* [HINT: Show that there are only finitely many preimages of a under F in the compact set $\{(x, v): |v| \leq 1\} \subset T(X)$. For if (x_i, v_i) are infinitely many preimages, pick a subsequence so that $x_i \longrightarrow x$, $v_i/|v_i| \longrightarrow w$. Now show that $df_x(w) = 0$.]

12. Whitney showed† that for maps of two-manifolds into $\mathbf{R}^3$, a typical cross cap looks like the map $(x, y) \longrightarrow (x, xy, y^2)$. Check that this is an immersion except at the origin. What does its image look like?

13. An open cover $\{V_\alpha\}$ of a manifold X is *locally finite* if each point of X possesses a neighborhood that intersects only finitely many of the sets V_α. Show that any open cover $\{U_\alpha\}$ admits a locally finite refinement $\{V_\alpha\}$. [HINT: Partition of unity.]

***14.** *Inverse Function Theorem Revisited.* Use a partition-of-unity technique to prove a noncompact version of Exercise 10, Section 3. Suppose that the derivative of $f: X \longrightarrow Y$ is an isomorphism whenever x lies in the submanifold $Z \subset X$, and assume that f maps Z diffeomorphically onto $f(Z)$. Prove that f maps a neighborhood of Z diffeomorphically onto a neighborhood of $f(Z)$. [*Outline:* Find local inverses $g_i: U_i \longrightarrow X$, where $\{U_i\}$ is a locally finite collection of open subsets of Y covering $f(Z)$. Define $W = \{y \in U_i : g_i(y) = g_j(y)$ whenever $y \in U_i \cap U_j\}$. The maps g_i "patch together" to define a smooth inverse $g: W \longrightarrow X$. Finish by proving that W contains an open neighborhood of $f(Z)$; this is where local finiteness is needed.]

15. *The Smooth Urysohn Theorem.* If A and B are disjoint, smooth, closed subsets of a manifold X, prove that there is a smooth function ϕ on X such that $0 \leq \phi \leq 1$ with $\phi = 0$ on A and $\phi = 1$ on B. [HINT: Partition of unity.]

†H. Whitney, "The General Type of Singularity of a Set of $2n - 1$ Functions of n Variables," *Duke Math. Journal,* **10** (1943), 161–172.